Sequential Quantum Cloning

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Not all unitary operations upon a set of qubits can be implemented by sequential interactions between each qubit and an ancillary system. We analyze the specific case of sequential quantum cloning, $1 \to M$, and prove that the minimal dimension D of the ancilla grows *linearly* with the number of clones M. In particular, we obtain $D = 2M$ for symmetric universal quantum cloning and $D = M + 1$ for symmetric phase-covariant cloning. Furthermore, we provide a recipe for the required ancilla-qubit interactions in each step of the sequential procedure for both cases.

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Multipartite entangled states stand up as the most versatile and powerful tool to perform information-processing protocols in Quantum Information Science [1]. They arise as an invaluable resource in tasks such as quantum computation [2, 3], quantum state teleportation [4], quantum communication [5] and dense coding [6]. As a result, the controllable generation of these states becomes a crucial issue in the quest for quantum-informational proposals. However, the generation of multipartite entangled states through single global unitary operations is, in general, an extremely difficult experimental task. In this sense, the sequential generation studied by Schön *et al.* [7], where at each step one qubit is allowed to interact with an ancilla, appears as the most promising avenue. The essence of this sequential scheme is the successive interaction of each qubit initialized in the standard state $|0\rangle$ with an ancilla of a suitable dimension D to generate the desired multiqubit state. In the last step, the qubit-ancilla interaction is chosen so as to decouple the final multiqubit entangled state from the auxiliary D-dimensional system, yielding [7]

$$|\Psi\rangle = \sum_{i_1\cdots i_n=0,1} \langle\varphi_F|V_{[n]}^{i_n}\cdots V_{[1]}^{i_1}|\varphi_I\rangle|i_1\cdots i_n\rangle. \quad (1)$$

Here, the $V_{[k]}^{i_k}$ are $D-$dimensional matrices arising from the isometries (unitaries) $V_{[k]} : \mathfrak{h}_A \otimes (|0\rangle) \to \mathfrak{h}_A \otimes \mathfrak{h}_{B_k}$, with $\mathfrak{h}_A = \mathbb{C}^D$ and $\mathfrak{h}_{B_k} = \mathbb{C}^2$ being the Hilbert spaces for the ancilla and the kth qubit, respectively, and where $|\varphi_I\rangle$ and $|\varphi_F\rangle$ denote the initial and final states of the ancilla, respectively. The state (1) is, indeed, a Matrix Product State (MPS) (cf. e.g. [8] and references therein), already present in spin chains [9], classical simulations of quantum entangled systems [10] and density-matrix renormalization group techniques [11]. Moreover, it was proven that any multiqubit MPS can be sequentially generated using the recipe of Ref. [7]. Notice that in this formalism, the mutual qubit-ancilla interaction in each step k completely determines the matrices $V_{[k]}^{i_k}$, $i_k = 0, 1$, whereas we enjoy some freedom to build such an interaction from a known $V_{[k]}^{i_k}$. This freedom stems from the fact that in the proposed scheme only the initial state $|0\rangle$ for each qubit is relevant.

In this letter, we consider the possibility of implementing quantum cloning based on a sequential protocol with the help of an ancillary system. This problem is certainly far from being an application of Ref. [7], given that the initial and final states are unknown. In this sense, any proposed strategy will be closer to the open problem of which global unitary operations (certainly not all of them) can be implemented through a sequential procedure. Despite the fundamental no-cloning theorem [12], stating the impossibility to exactly clone an unknown quantum state, there exists several cloning techniques with a given optimal fidelity [13]. These procedures differ either from the initial set of states to be cloned or from symmetry considerations. In general, an optimality condition of the cloning procedure is obtained via the maximization of the fidelity between the original qubit and each final clone state. We will show how to perform sequentially both the universal symmetric [14, 15] and the economical phase-covariant symmetric quantum cloning [16, 17] from one qubit to M clones. In the first case, a global unitary evolution transforms *any input state* $|\psi\rangle$ in a set of M clones whose individual reduced states ρ_{out} carry maximal fidelity with respect to $|\psi\rangle$: $F_{1,M} = \frac{2M+1}{3M}$. This cloning procedure is fully described by the evolution

$$|\psi\rangle \otimes |B\rangle \to |GM_M(\psi)\rangle \equiv$$
$$\equiv \sum_{j=0}^{M-1} \alpha_j |(M-j)\psi, j\psi^{\perp}\rangle_S \otimes |(M-j-1)\psi^*, j\psi^{*\perp}\rangle_S, \quad (2)$$

where $|B\rangle$ denotes the initial blank state, $\alpha_j = \sqrt{\frac{2(M-j)}{M(M+1)}}$ and $|(M-j)\psi, j\psi^{\perp}\rangle_S$ denotes the normalized completely symmetric state with $(M-j)$ qubits in state ϕ and j

qubits in state $\phi^\perp$. As a relevant feature it must be noticed that the presence of $M-1$ additional so-called anticlones is necessary in order to perform this cloning procedure with the optimal fidelity. The anticlone state ψ^* refers to the fact that they transform under rotations as the complex conjugate representation. For concreteness sake we have chosen $|\psi^*\rangle = \cos\theta/2|1\rangle + e^{-i\phi}\sin\theta/2|0\rangle$ in coincidence with the seminal paper by Bužek and Hillery [14], whereas $|\psi\rangle = \cos\theta/2|0\rangle + e^{i\phi}\sin\theta/2|1\rangle$. In the second case, motivated by quantum cryptoanalysis, the goal is to clone only those states belonging to the equatorial plane of the Bloch sphere, i.e. those such that $\theta = \pi/2$. Furthermore, we have only focused upon the cases where no anticlones are needed (hence the term economical). Under this assumption, imposing the purity of the joint state, the number of clones M must be odd [16]. The cloning evolution is now given by

$$|\psi\rangle \otimes |B\rangle \to \frac{1}{\sqrt{2}}\left[|(k+1)0, k1\rangle_S + e^{i\phi}|k0,(k+1)1\rangle_S\right], \tag{3}$$

where $k = (M-1)/2$ and where we have followed the same convention as above. In order to employ the sequential ancilla-qubit device as a quantum cloning machine we will firstly elucidate the minimal dimension required for the ancilla. The basic idea is to express the final states (2) and (3) in its MPS form, following the recipe of Ref. [10], $|\Phi\rangle = \sum_{i_1\ldots i_N} c_{i_1\ldots i_N}|i_1\ldots i_N\rangle$, with

$$c_{i_1\ldots i_N} = \sum_{\alpha_1\ldots\alpha_{n-1}} \Gamma[1]^{i_1}_{\alpha_1}\lambda[1]_{\alpha_1}\Gamma[2]^{i_2}_{\alpha_1\alpha_2}\lambda[2]_{\alpha_2}\ldots\Gamma[n]^{i_n}_{\alpha_{n-1}}. \tag{4}$$

We identify the matrices $V^{i_k}_{[k]}$ by matching indices in expressions (1) and (4). The indices α_j run from 1 to χ, where $\chi = \max_{\mathcal{P}}\{\chi_{\mathcal{P}}\}$, $\chi_{\mathcal{P}}$ denoting the rank of the reduced density matrix $\rho_{\mathcal{P}}$ for the bipartite partition $\mathcal{P}$ of the composite system [10]. To clone an arbitrary input qubit state $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$, we exploit linearity and determine the minimal dimension $D_{0,1}$ of the ancilla to perform the cloning for the state $|0\rangle$ and then similarly for the state $|1\rangle$. Then we combine both results in a single ancilla to obtain the desired minimal dimension D of the ancilla. Let us focus upon the symmetric universal cloning of $|0\rangle$. To determine the minimal dimension D_0 of the ancilla we need to compute χ, which can be done without the exact MPS expression for the state.

Proposition 1. *Let $|\psi\rangle$ and $|\phi\rangle$ be multipartite states of the same system related through an invertible local operator $F_{\mathcal{P}} = F_A \otimes F_B$ for the partition $\mathcal{P} = A|B$, such that*

$$|\psi\rangle = F_{\mathcal{P}}|\phi\rangle. \tag{5}$$

Then $\chi_{\mathcal{P}}(\phi) = \chi_{\mathcal{P}}(\psi)$.

Proof. Recalling that the rank of $\rho_{\mathcal{P}}$ coincides with the rank of the coefficient matrix $C_{\mathcal{P}}$ of the corresponding state for that partition $\mathcal{P}$, the application of the invertible local operator $F_{\mathcal{P}}$ amounts to changing the local basis of each part A and B in which the coefficient matrix is expressed. Since the rank is invariant under local changes of basis, we will have $\mathrm{r}(C_{\mathcal{P}}(\phi)) = \mathrm{r}(C_{\mathcal{P}}(\psi))$ for the bipartite partition $\mathcal{P}$. Hence $\chi_{\mathcal{P}}(\phi) = \chi_{\mathcal{P}}(\psi)$. □

Proposition 2. *Let $C_{M|M-1}$ be the coefficient matrix of the state (2) for the partition $M|M-1$. Then*

$$\mathrm{r}(C_{M|M-1}) = M. \tag{6}$$

Proof. In virtue of the preceding proposition and using the invertible local operator $\mathcal{S}_M \otimes \mathcal{S}_{M-1}$, where $\mathcal{S}_K$ denotes the normalized symmetrizing operator for K qubits, we only need to compute the rank of the coefficient matrix of

$$\sum_{j=0}^{M-1}\alpha_j|(M-j)0, j1\rangle \otimes |(M-j-1)1, j0\rangle, \tag{7}$$

where the states are no longer completely symmetrized. Given the orthonormality of the involved states and the number of different components (M), it is clear that there are only M different rows, whereas the rest are all null, i.e. $\mathrm{r}(C_{M|M-1}) = M$. □

Proposition 3. *Let $C_{k|2M-1-k}$ be the coefficient matrix of state (2) for the partition $k|2M-k-1$, where $k = 1, 2, \ldots, 2M-2$. Then*

$$\mathrm{r}(C_{k|2M-1-k}) \le \mathrm{r}(C_{M|M-1}) \qquad \forall k. \tag{8}$$

Proof. The key point is to realize that the matrices $C_{k|2M-1-k}$ are obtained from $C_{M|M-1}$ by appropriately adjoining rows or columns to make them longer. From the preceding proof it is clear that there are only M different rows in $C_{M|M-1}$, the rest being all null, thus the reordering procedure to build the other matrices cannot increase the former rank. Hence the stated result. □

With all these propositions it elementarily follows that $\chi = M$, i.e. that the minimal dimension D_0 to clone the $|0\rangle$ state is $D_0 = M$, namely the number of clones to produce. Repeating the same argument for the initial state $|1\rangle$ we also conclude that the minimal dimension of the ancilla to clone the $|1\rangle$ state is $D_1 = M$, as expected. Now we must combine both results to find D for an arbitrary unknown state $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$. It is a wrong guessing to think that it should also be $D = M$ and, consequently, a different scheme must be given. The MPS expression of (2) for the original state $|0\rangle$ determines the D-dimensional matrices $V^{i_k}_{0[k]}$, whereas the corresponding MPS expression for the original state $|1\rangle$ determines $V^{i_k}_{1[k]}$,

$$|GM_M(0)\rangle = \sum_{i_1\ldots i_n=0,1}\langle\varphi^{(0)}_F|V^{i_n}_{0[n]}\ldots V^{i_1}_{0[1]}|0\rangle_D|i_1\ldots i_n\rangle,$$

$$|GM_M(1)\rangle = \sum_{i_1 \ldots i_n = 0,1} \langle \varphi_F^{(1)} | V_{1[n]}^{i_n} \ldots V_{1[1]}^{i_1} |0\rangle_D |i_1 \ldots i_n\rangle. \tag{9}$$

Here, $|\varphi_F^{(0)}\rangle$ and $|\varphi_F^{(1)}\rangle$ can be calculated explicitly and will play an important role below.

We propose now to double the dimension of the ancilla, $\mathbb{C}^D \to \mathbb{C}^2 \otimes \mathbb{C}^D$, in order to implement a deterministic protocol of sequential quantum cloning.

Protocol 1. *i. Encode the unknown state $|\psi\rangle$ in the initial ancilla state $|\varphi_I\rangle = |\psi\rangle \otimes |0\rangle_D$.*

ii. Allow each qubit k to interact with the ancilla according to the $2D$-dimensional isometries $V_{[k]}^{i_k} = |0\rangle\langle 0| \otimes V_{0[k]}^{i_k} + |1\rangle\langle 1| \otimes V_{1[k]}^{i_k}$.

iii. Perform a generalized Hadamard transformation upon the ancilla

$$|0\rangle \otimes |\varphi_F^{(0)}\rangle \to \frac{1}{\sqrt{2}} \left[|0\rangle \otimes |\varphi_F^{(0)}\rangle + |1\rangle \otimes |\varphi_F^{(1)}\rangle \right],$$
$$|1\rangle \otimes |\varphi_F^{(1)}\rangle \to \frac{1}{\sqrt{2}} \left[|0\rangle \otimes |\varphi_F^{(0)}\rangle - |1\rangle \otimes |\varphi_F^{(1)}\rangle \right]. \tag{10}$$

Note that the choice $\mathbb{C}^D \to \mathbb{C}^2 \otimes \mathbb{C}^D$ (based on pedagogical reasons) could be changed, equivalently, to $\mathbb{C}^D \to \mathbb{C}^{2D}$. In this way, Eq. (10) would not display entangled states but simple linear superpositions.

iv. Perform a measurement upon the ancilla in the local basis $\{|0\rangle \otimes |\varphi_F^{(0)}\rangle, |1\rangle \otimes |\varphi_F^{(1)}\rangle\}$.

v. If the result is $|0\rangle \otimes |\varphi_F^{(0)}\rangle$ (which happens with probability 1/2), the qubits are already in the desired state; if the result is $|1\rangle \otimes |\varphi_F^{(1)}\rangle$ (probability 1/2), perform a local π-phase gate upon each qubit, then they will end up in the desired state.

Proof. After the first two steps, the joint state of the ancilla and the qubits is $\alpha \left(|0\rangle \otimes |\varphi_F^{(0)}\rangle \right) \otimes |GM_M(0)\rangle + \beta \left(|1\rangle \otimes |\varphi_F^{(1)}\rangle \right) |GM_M(1)\rangle$, where originally $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$. After the Hadamard rotation, step (iii), this state becomes

$$\frac{1}{\sqrt{2}} \left(|0\rangle \otimes |\varphi_F^{(0)}\rangle \right) \otimes [\alpha |GM_M(0)\rangle + \beta |GM_M(1)\rangle] + $$
$$+ \frac{1}{\sqrt{2}} \left(|1\rangle \otimes |\varphi_F^{(1)}\rangle \right) \otimes [\alpha |GM_M(0)\rangle - \beta |GM_M(1)\rangle] .$$

The remaining steps follow immediately from this expresion and from linearity [15]. □

Notice that despite the measurement process in step (iv), the final desired state is obtained with probability 1. In summary, the minimal dimension D of the ancilla for cloning M qubits is $D = 2 \times M$, i.e., it grows linearly with the number of clones even if the dimension of their space grows exponentially (2^M).

	$k=0$	$k=1$
$\left[V_{0[1]}^k\right]_{ij} =$	$\begin{cases} \delta_{ij}\mathcal{C}(2-i,i-1) & 1 \le i,j \le 2 \\ \frac{1}{\sqrt{2}}\delta_{ij} & \text{otherwise} \end{cases}$	$\begin{cases} \delta_{i,3-j}\mathcal{C}(2-i,i-1) & 1 \le i,j \le 2 \\ \frac{1}{\sqrt{2}}\delta_{ij} & \text{otherwise} \end{cases}$
$\left[V_{0[n]}^k\right]_{ij} =$	$\begin{cases} \delta_{ij}\frac{\mathcal{C}(n+1-i,i-1)}{\mathcal{C}(n-i,i-1)} & 1 \le i,j \le n \\ \frac{1}{\sqrt{2}}\delta_{ij} & \text{otherwise} \end{cases}$	$\begin{cases} \frac{1}{\sqrt{2}} & i=1, j=n+1 \\ \delta_{i,j+1}\frac{\mathcal{C}(n-j,j)}{\mathcal{C}(n-j,j-1)} & 2 \le i \le n+1, 1 \le j \le n \\ \frac{1}{\sqrt{2}}\delta_{ij} & \text{otherwise} \end{cases}$
$\left[V_{0[M]}^k\right]_{ij} =$	$\begin{cases} \delta_{ij}\frac{\alpha_{i-1}}{\mathcal{C}(M-i,i-1)\sqrt{\binom{M}{i-1}}} & 1 \le i,j \le M \end{cases}$	$\begin{cases} \delta_{i,j+1}\frac{\alpha_j}{\mathcal{C}(M-j,j-1)\sqrt{\binom{M}{j}}} & 1 \le i,j \le M \end{cases}$
$\left[V_{0[M+n]}^k\right]_{ij} =$	$\begin{cases} \delta_{i,j-1}\sqrt{\frac{i}{M-n}} & \begin{cases} 1 \le i \le M-n \\ 2 \le j \le M-n+1 \end{cases} \\ \frac{1}{\sqrt{2}} & i=M-n+1, j=1 \\ \frac{1}{\sqrt{2}}\delta_{ij} & \text{otherwise} \end{cases}$	$\begin{cases} \delta_{ij}\sqrt{\frac{M-n+1-i}{M-n}} & 1 \le i,j \le M-n \\ \frac{1}{\sqrt{2}}\delta_{ij} & \text{otherwise} \end{cases}$

TABLE I: Matrices for the universal symmetric cloning protocol.

It can be checked straightforwardly that if one had to clone a d-dimensional system, the minimal dimension for the ancilla would be $D = d \times M$, an obvious generalization of the preceding argument.

For the symmetric phase-covariant cloning, the same arguments can be reproduced. For example, the first term on the r.h.s. of Eq. (3) can be cast in the form of the state in Eq. (2)

$$|(k+1)0, k1\rangle_S = \\ = \sum_{j=0}^{k} \gamma_j |(k+1-j)0, j1\rangle_S \otimes |(k-j)1, j0\rangle_S, \quad (11)$$

where $\gamma_j \neq 0$ for all j, and similarly for the second term. Thus for symmetric phase-covariant cloning the minimal dimension for the ancilla is $D = 2 \times (k+1) = 2 \times \frac{M+1}{2} = M+1$. We see that the dimension of the ancilla D also grows linearly with the number of clones, although it is now lesser than above. This is a direct consequence of the reduction in the set of possible original states to clone.

For the symmetric universal cloning we give in detail in Table I the $2D-$dimensional matrices $V_{[k]}^{i_k}$ driving us to a concrete sequential scheme, and where $\mathcal{C}(i,j) := \sqrt{\frac{1}{\binom{i+j}{i}} \sum_{k=j}^{M-1} |\alpha_k|^2 \frac{\binom{M-k}{i}\binom{k}{j}}{\binom{M}{i+j}}}$, $\binom{p}{q} = 0$ if $q > p$ and $1 < n \leq M-1$. Furthermore, we also have $V_{1[k]}^{i_k} = V_{0[k]}^{\bar{i}_k}$, where by $\bar{i}$ we indicate $\bar{i} := i \oplus 1$. They coincide also with the ones for the symmetric phase-covariant cloning just by doing the substitutions $M \to \frac{M+1}{2}$ and $\alpha_j \to \gamma_j := \sqrt{\frac{\binom{k+1}{k+1-j}\binom{k}{j}}{\binom{2k+1}{k+1}}}$.

It can be readily verified that the minimal dimension for the ancilla is $2 \times M$. When sequentially applying these matrices to the initial state $|\varphi_I\rangle$ of the ancilla, one can check, as expected, that if we were to stop at the Mth step, the M clones would have already been produced with the desired properties, although in a highly entangled state with the ancilla. To arrive at a final uncoupled state, the remaining $M-1$ anticlones must be operated upon by the ancilla.

In conclusion, we have shown how to reproduce sequentially both the symmetric universal and symmetric phase-covariant cloning operations. For the universal cloning we have proved that the minimal dimension for the ancilla should be $D = 2M$, where M denotes the number of clones, thus showing a linear dependence. The original state must be encoded in a $2D-$dimensional state. For the phase-covariant case, the required dimension D of the ancilla can be reduced to $D = M+1$. In both cases, the ancilla ends up uncoupled to the qubits. Along similar lines, this sequential cloning protocol can be adapted to other proposals, such as asymmetric universal quantum cloning machines or other state-dependent protocols. This procedure can have notable experimental interest, since it provides a systematic method to furnish any multiqubit state using only sequential two-system (qubit-ancilla) operations.

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